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LETTER TO THE EDITOR

q-magnetism at roots of unity

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Abstract. We study the thermodynamic properties of a family of integrable 1D spin chain Hamiltonians associated with quantum groups at roots of unity. These Hamiltonians depend for each primitive root of unity on a parameter s which plays the role of a continuous spin. The model exhibits ferromagnetism even though the interaction involved is between nearest neighbours. The latter phenomenon is interpreted as a genuine quantum group effect with no 'classical' analogue. The discussion of conformal properties is given.

After Heisenberg [1], spin is the key word for understanding the magnetic properties of metals. In one spatial dimension we have many exactly solvable models, which can be treated by means of the Bethe ansatz technique [2]. These models can be used to deepen our intuition on such non-trivial subjects as magnetism. Quantum groups [3] provide the mathematical ground for studying integrable one-dimensional spin chains. Moreover, the different integrable generalizations of the original $S = \frac{1}{2}$ Heisenberg model are associated in one to one fashion with the different irreps of $U_q(SL(2))$ where the deformation parameter q is related to the anisotropy of the chain.

Heisenberg's ideas of magnetism can be extended naturally in the context of quantum groups, in a sense that the rotational group SU(2) is replaced by $U_q(SL(2))$. For generic q this replacement is not essential, just because the finite-dimensional irreps of SU(2) and the ones of its quantum deformation are the same. If we want a typical signal of the effect of defining the spin variables by finite-dimensional irreps of $U_q(SL(2))$, 'q-magnetism', we need to work in a very special regime, a q root of unity, where we have finite-dimensional irreps of $U_q(SL(2))$ without 'classical' $(q = 1)$ analogue [4].

In this letter we start a systematic study of the magnetic properties of one-dimensional spin chains using non-regular finite-dimensional irreps of $U_q(SL(2))$ at roots of unity. The main new phenomena we find, concerning the magnetic properties, is ferromagnetism, i.e. a disordered ground state with non-vanishing magnetization. This kind of behaviour is known in systems possessing complex topology of interaction [5], while here the appearance of this phenomena is directly tied with the special irreps used to define the spin variables of the chain.

The quantum group $U_q(SL(2))$ with $q = \epsilon$, $\epsilon^N = 1$, is generated by the operators E, F and $K = \epsilon^{2s_z}$. The peculiar thing about ϵ being a root of unity is that $E^{N'}$, $F^{N'}$ and $K^{N'}$ are central elements (where $N' = N$ if N is odd and $N' = N/2$ if

N is even). These central elements, together with the Casimir, label the irreps of $U_q(SL(2))$. Regular irreps, which are the q-deformations of the usual integer and half-integers spin representations, satisfy $E^{N'} = F^{N'} = 0$ and $K^{N'} = \pm 1$. Nilpotent irreps of $U_\epsilon(SL(2))$ are a slight generalization of the regular ones, in the sense that the generator K takes on the generic value ϵ^{2s}, where s is our 'continuous' spin. The dimension of these nilpotent irreps is always N'.

The 'nilpotent' spin chain Hamiltonian is defined in the standard way as

$$\begin{aligned} H(s) &= -\mathrm{i}I \frac{\partial \ln t(u,s)}{\partial u}\Big|_{u=0} \\ t(u,s) &= \lim_{s'\to s} \mathrm{tr}_s\left(R_L^{ss'}(u) R_{L-1}^{ss'}(u) \cdots R_1^{ss'}(u)\right) \end{aligned} \tag{1}$$

where I is an overall coupling constant and L is the total number of sites. The quantum R-matrix $R^{ss'}(u)$ intertwining two nilpotent irreps of $U_q(\widehat{SL(2)})$ is given best by its non-vanishing matrix elements [6]

$$\begin{aligned} R^{ss'}(u)_{r_1,r_2}^{l,r_1+r_2-l} &= \frac{1}{\prod_{j=0}^{r_1+r_2-1}(e^u k_1 k_2 \epsilon^{-j} - e^{-u}\epsilon^j)} \\ &\times \sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2} \begin{bmatrix} r_1 \\ l_1 \end{bmatrix} \begin{bmatrix} r_2 \\ l_2 \end{bmatrix} \frac{[l]![r_2-l_2]!}{[r_1+l_2]![r_2]!}(\epsilon-\epsilon^{-1})^{r_1-l_1+l_2} \\ &\times \prod_{j=r_1}^{r_1+l_2-1} d_j(k_1) \prod_{j=l_1+l_2}^{r_1+l_2-1} d_j(k_1) \prod_{j=r_2-l_2}^{r_2-1} d_j(k_2) \prod_{j=r_2-l_2}^{r_1+r_2-l-1} d_j(k_2) \\ &\times k_1^{l_2} k_2^{r_1-l_1} \prod_{j=0}^{r_2-l_2-1} (e^u k_2 \epsilon^{-j} - e^{-u} k_1 \epsilon^j) \\ &\times \prod_{j=0}^{l_1-1} (e^u k_1 \epsilon^{-j+r_2-l_2} - e^{-u} k_2 \epsilon^{j+l_2-r_2}) \end{aligned} \tag{1a}$$

where r_1, r_2, l and $r_1 + r_2 - l = 0, 1, \ldots, N'$ and

$$k_1 = \epsilon^{2s} \qquad k_2 = \epsilon^{2s'} \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad d_j^2(k) = [j+1]\frac{k\epsilon^{-j} - k^{-1}\epsilon^j}{\epsilon - \epsilon^{-1}} \tag{1b}$$

with the following conventions: (a) whenever, in the above products, the upper index is less than the lower index the result is one; (b) the constraint $l_1 + l_2 = l$ must be used to carry out the summation.

The explicit expression for this type of Hamiltonian ($N = 3$) can be found in [8]. As shown there, for the special case $N = 3$ and $s = 1$ the Hamiltonian (1) coincides with the Fateev–Zamolodchikov Hamiltonian [7] with the anisotropy fixed by $q = e^{2\pi \mathrm{i}/3}$. The hermiticity conditions on the Hamiltonian $H(s)$ are given by

$$v_s \sin\gamma k \sin\gamma(2s - k + 1) > 0 \qquad k = 1, 2, \ldots, N' - 1 \tag{2}$$

where $q = e^{i\gamma}$ and $v_s = \pm 1$ is the spin parity. Equation (2) is equivalent to the condition $E^\dagger = v_s F$ for the corresponding nilpotent s-irrep. The hermiticity regions that follow from (2) (for $\epsilon = e^{2\pi i/N}$) are

$$\begin{aligned} &\text{N even:} \qquad \frac{1}{2} - \frac{1}{p_0} + \frac{1-v_s}{4} < \frac{s}{p_0} < \frac{1}{2} + \frac{1-v_s}{4} \\ &\text{N odd:} \qquad \frac{1}{2} - \frac{3}{4p_0} + \frac{1-v_s}{4} < \frac{s}{p_0} < \frac{1}{2} - \frac{1}{4p_0} + \frac{1-v_s}{4} \end{aligned} \tag{3}$$

where $p_0 = N/2$. In the following, we shall consider $N > 4$ (N even) and $N > 3$ (N odd). In the trivial case $N = 4$, the Hamiltonian (1) is essentially that of the XX model in a magnetic field. The case $N = 3$ [8] requires special treatment which will be given elsewhere [9].

Notice that, for N even, the middle point of both spin intervals ($v_s = \pm 1$) corresponds to a regular integer or half-integer spin. For N odd only the interval of negative parity contains such a point. For all these middle points s_0's the corresponding Hamiltonians $H(s_0)$ are identical to higher spin XXZ models with anisotropy $\gamma = 2\pi/N$. It is interesting to observe that $2s_0 + 1$ is not a Takahashi number [10]. Apparently for that reason, Kirillov and Reshetikhin [11] do not consider this case in their, otherwise, general analysis. On the other hand, Babujian and Tsvelick [12] have considered one of these points ($s = (N-2)/4$ for N even). However, we do not believe that their results are correct concerning this point.

The Hamiltonian (1) can be diagonalized by means of the standard Bethe ansatz [6]. The Bethe ansatz equations read, in our case

$$\left[\frac{\sinh\frac{\gamma}{2}(\lambda_j + 2is)}{\sinh\frac{\gamma}{2}(\lambda_j - 2is)}\right]^L = -\prod_{k=1}^{M} \frac{\sinh\frac{\gamma}{2}(\lambda_j - \lambda_k + 2i)}{\sinh\frac{\gamma}{2}(\lambda_j - \lambda_k - 2i)} \tag{4}$$

with the energy eigenvalues given by

$$E_M = -\sum_{k=1}^{M} \frac{I \sin 2\gamma s}{\sinh[\frac{\gamma}{2}(\lambda_k + 2is)] \sinh[\frac{\gamma}{2}(\lambda_k - 2is)]} \tag{5}$$

where s is our 'generic' spin, subject only to hermiticity requirements (3).

To solve (4) we will use the string hypothesis (SH) [13]

$$\lambda_l^n = \lambda_c^n + i[n + 1 - 2l + (\pi/2\gamma)(1 - v_s v_n)] \tag{6}$$

where $l = 1, \ldots, n$ and λ_c^n is the real-valued centre of the string of length n and parity $v_n = \pm 1$. It can be proven that the allowed strings are determined by the Takahashi condition [10]

$$v_n \sin\gamma(n-l) \sin\gamma l > 0 \qquad l = 1, 2, \ldots, n-1 \tag{7}$$

whenever the hermiticity condition (2) holds true. Strictly speaking, SH is legitimate only if the number of BA roots is much smaller than the number of sites. However, it has been shown [14] that the SH can be safely used for the non-zero magnetic field or temperature.

Using the 'Takahashi zone' terminology, we have for the allowed strings (n_j, v_j)

$$N \text{ even:} \quad \begin{cases} \text{0-zone} & n_j = j & v_{n_j} = +1 & 1 \leqslant j \leqslant \nu - 1 \\ \text{1-zone} & n_\nu = 1 & v_\nu = -1 & j = \nu \end{cases}$$

$$N \text{ odd:} \quad \begin{cases} \text{0-zone} & n_j = j & v_j = +1 & 1 \leqslant j \leqslant \nu - 1 \\ \text{1-zone} & \begin{cases} n_\nu = 1 \\ n_{\nu+1} = \nu + 1 \end{cases} & \begin{matrix} v_\nu = -1 \\ v_{\nu+1} = +1 \end{matrix} & \begin{matrix} j = \nu \\ j = \nu + 1 \end{matrix} \\ \text{2-zone} & n_{\nu+2} = \nu & v_{\nu+2} = +1 & j = \nu + 2 \end{cases} \tag{8}$$

where $\nu = \frac{1}{2}N$ for N even and $\nu = \frac{1}{2}(N-1)$ for N odd.

In the thermodynamic limit equations (4) become

$$\tilde{a}_j = (-1)^{r_j}(\rho_j + \rho_j^h) + \sum_k T_{jk} \star \rho_k \tag{9}$$

where $\rho_j(\rho_j^h)$ is the density of j-strings (j-holes) and $(-1)^{r_j}$ is the sign of $\tilde{a}_j(\lambda)$ and '$\star$' stands for convolution. The Fourier transforms of the functions which appear in (9) are given by

$$\begin{aligned} \hat{T}_{jk} &= g(\omega; |n_j - n_k|; v_{n_j} v_{n_k}) + g(\omega; |n_j + n_k|, v_{n_j} v_{n_k}) \\ &\quad + 2(1 - \delta_{1,\min(n_j,n_k)}) \sum_{l=1}^{\min(n_j,n_k)-1} g(\omega; |n_j - n_k| + 2l; v_{n_j} v_{n_k}) \\ \hat{a}_j &= \sum_{l=0}^{n_j - 1} g(\omega; 2s + 1 - n_j + 2l, v_s v_{n_j}) \\ g(\omega; n; v) &= -\frac{\sinh 2p_0\omega((n/2p_0 + (1-v)/4))}{\sinh p_0\omega} \end{aligned} \tag{10}$$

where $((\cdots))$ is the Dedekind function.

Following Yang and Yang [15] we minimize the free energy $F = E - TS$ to obtain

$$\begin{aligned} \frac{F}{L} &= -T \sum_j \int_{-\infty}^{+\infty} \mathrm{d}\lambda |\tilde{a}_j(\lambda)| \ln(1 + \eta_j^{-1}) \\ \ln \eta_j &= -\frac{4p_0 I}{T} \tilde{a}_j + \sum_k (-1)^{r_k} T_{jk} \star \ln(1 + \eta_k^{-1}) \\ \eta_j &= \exp\left(\frac{\epsilon_j(\lambda)}{T}\right) = \frac{\rho_j^h(\lambda)}{\rho_j(\lambda)}. \end{aligned} \tag{11}$$

In the $T = 0$ limit we obtain results for the ground state and the spectrum of excitations as given in table 1.

We observe that classification of the strings, given in table 1, is independent of the value of the spin s, as long as it belongs to the hermiticity regions (3). A comparison of the spectrum given above with that of [11] shows that they are quite different.

Table 1. Results for ground state and spectrum of excitations in the $T = 0$ limit. The entries refer to the label j of the strings (n_j, v_j).

N	I	Ground state strings	Positive energy strings	Zero energy strings
even	> 0	$\nu - 1$	ν	the rest
even	< 0	ν	the rest	none
odd	> 0	$\nu + 2$	$\nu, \nu + 1$	the rest
odd	< 0	$\nu, \nu + 1$	the rest	none

Interestingly enough, there is only one kind of string filling the Dirac sea (except for the case of N odd and $I < 0$). This will be important when we discuss the conformal properties of our models.

It is appropriate to point out that zero energy strings, contributing neither to energy nor momentum, play a vital role in the S-matrix calculations and in labelling the degenerate states. In fact, they provide quantum numbers describing spin-1/2 and parafermionic degrees of freedom. The origin of this phenomenon can be traced back to the symmetries of the Hamiltonian (1) which appear only in the infinite volume limit. Finally, we comment that energies of physical excitations show remarkable dependence on s (our generic spin), as illustrated in table 2.

Table 2. Energies of holes in distribution of ground state strings.

$\widehat{\epsilon}_j(w) = \int \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}w\lambda} \epsilon_j(\lambda)$	N even	N odd
$I > 0$	$\widehat{\epsilon}_{\nu_1 - 1}(w) = 4p_0 I \frac{\sinh w(2s - p_0 \frac{1}{2}(3 - v_s))}{\sinh 2w}$	$\widehat{\epsilon}_{\nu_1 + 2}(w) = 8p_0 I \frac{\cosh \frac{w}{2} \sinh w(2s - p_0 \frac{1}{2}(3 - v_s))}{\sinh 2w}$
$I < 0$	$\widehat{\epsilon}_{\nu_1}(w) = 2p_0 I \frac{\sinh w(2s - p_0 \frac{1}{2}(1 - v_s))}{\sinh w \cosh w(p_0 - 1)}$	$\widehat{\epsilon}_{\nu_1}(w) = 2p_0 I \frac{\sinh w(p_0 - \frac{3}{2}) \cosh w(2s + 1 - p_0 \frac{1}{2}(3 - v_s))}{\sinh w \cosh \frac{w}{2} \cosh w(p_0 - 1)}$ $\widehat{\epsilon}_{\nu_1 + 1}(w) = 4p_0 I \frac{\sinh w(2s + 3/2 - p_0 \frac{1}{2}(3 - v_s))}{\sinh w}$

The $T \to \infty$ limit of equations (11) provides a justification of the SH. In fact, we get $\lim_{T \to \infty} F/TL = -\ln N'$, which implies that the total number of states is correctly given by $(N')^L$.

Next we move on to compute entropy S

$$\frac{S}{L} = \sum_j \int \mathrm{d}\lambda\, \rho_j(\lambda)[(1 + \eta_j)\ln(1 + \eta_j) - \eta_j \ln \eta_j]. \tag{12}$$

Making use of equations (11), we obtain in the low temperature limit

$$\text{N even:} \qquad \frac{S}{L} = \begin{cases} \dfrac{2T\pi}{6v^s}\left[3 - \dfrac{6}{\nu + 1}\right] & I > 0 \\ \dfrac{2T\pi}{6v^s} & I < 0 \end{cases} \tag{13a}$$

N odd:
$$\frac{S}{L} = \begin{cases} \dfrac{2T\pi}{6v^{s}}\left[3 - \dfrac{6}{\nu+2}\right] & I > 0 \\ \dfrac{4T}{\pi}\left[\dfrac{1}{v_1^{s}}L\left(\dfrac{\nu}{2\nu+1}\right) + \dfrac{1}{v_2^{s}}L\left(\dfrac{\nu+1}{2\nu+1}\right)\right] & I < 0 \end{cases} \tag{13b}$$

where v^{s}, v_1^{s} and v_2^{s} are speeds of sound

$$v^{s} = \tfrac{1}{2}N|I| \qquad v_1^{s} = \frac{\frac{1}{2}N|I|}{\frac{1}{2}N-1} \qquad v_2^{s} = N|I| \tag{14}$$

and $L(x)$ is the dilogarithmic Roger function [16]. Notice that for N odd and $I < 0$ we have two different speeds of sound. For the remaining cases there is only one speed of sound so that the underlying CFT has a central extension c given by

$$\frac{\partial S}{\partial T} \equiv -\frac{\partial^2 F}{\partial T^2} = \frac{\pi c}{3v^{s}}. \tag{15}$$

From equation (13) we get

$$I > 0 \qquad c = \frac{3s_{\text{eff}}}{s_{\text{eff}}+1}$$
$$I < 0 \qquad c = 1 \qquad \text{for } N \text{ even}$$

where

$$s_{\text{eff}} = \begin{cases} \frac{1}{4}(N-2) & N \text{ even} \\ \frac{1}{4}(N-1) & N \text{ odd.} \end{cases}$$

When N is odd and $I < 0$ there are two different strings filling the ground state and two different speeds of sound. This fact indicates that rotational invariance is broken which, in turn, implies that we do not have a full conformal invariance. This situation has already been discussed in the literature [17], where a broken CFT (in the sense given above) can be viewed as a sum of two independent CFTs. In our case, we have not been able to identify any of the broken pieces with reasonable CFT.

Finally, we present our results for the magnetization of the ground state at $T = 0$ which is defined as

$$M = \frac{s^{z}}{L} = s - \sum_{j \in \text{ground state}} n_j \int \rho_j(\lambda)\,\mathrm{d}\lambda. \tag{16}$$

The results are collected in table 3.

Table 3. Results for magnetization of ground state at $T = 0$.

N	I	M
even	>0	$M = \frac{1}{2}N[s - (\frac{1}{2}N - 1)\frac{1}{4}(3 - v_s)]$
even	<0	$M = \frac{1}{2}N(\frac{1}{4}(1 - v_s))$
odd	>0	$M = N[s - (\frac{1}{2}(N-1))\frac{1}{4}(3 - v_s)]$
odd	<0	$M = -N[s + 1 - (\frac{1}{2}(N+1))\frac{1}{4}(3 - v_s)]$

From table 3, we see that for generic s (subject to hermiticity condition (3)) the ground state exhibits ferromagnetic behaviour. More precisely, when spin s takes on values different from integer or half-integer then magnetization is non-null. This non-vanishing M is produced *internally by representation itself*.

To summarize: q being a root of unity made it possible to depart from regular representations and this, in turn, led to the interesting phenomenon of ferromagnetism for a system governed by a local (nearest-neighbour interaction) Hamiltonian. Finally, we comment that contrary to the case where non-zero M is produced by external magnetic field, we have infinite Fermi band for the ground state strings $[\epsilon_{\mathrm{gr.\,state}}(\pm\infty) = 0]$ and all our excitations are massless. Thus, all conformal degrees of freedom are preserved and central charge does not depend on s, as can be seen from (15).

In future publications we hope to report on our study of magnetic properties of the model as well as on the further analysis of conformal properties and to present our study of scattering matrices along with quantum numbers of low-lying excitations. The details of the results presented here will be given elsewhere [9].

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Note added in proof. After this letter was submitted for publication it was brought to our attention by the referee that ferromagnetism also appears in the context of the Perk–Schultz model, as discussed in [18].

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